



*Citation for published version:*

Rees, DAS 2015, 'On convective boundary layer flows of a Bingham fluid in a porous medium', International Journal of Heat and Mass Transfer, vol. 82, pp. 206-212.  
<https://doi.org/10.1016/j.ijheatmasstransfer.2014.11.038>

*DOI:*

[10.1016/j.ijheatmasstransfer.2014.11.038](https://doi.org/10.1016/j.ijheatmasstransfer.2014.11.038)

*Publication date:*

2015

*Document Version*

Early version, also known as pre-print

[Link to publication](#)

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# On Convective Boundary Layer Flows of a Bingham Fluid in a Porous Medium

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**Abstract** In this short paper we consider the state-of-the-art with regard to convective boundary layer flows of yield-stress fluids in a porous medium. About a dozen papers have been published on the topic in the last 15 years or so and each has presented a leading order boundary layer theory. For natural convection boundary layers of such fluids, the streamwise velocity field is confined to the boundary layer region but it is also delimited by a yield surface at which there is a precise balance between the yield stress and the buoyancy force. The aim of the present paper is to examine whether such boundary layer flows can exist in practice. We draw on a rigorous boundary layer theory formulated in terms of an asymptotically large Darcy-Rayleigh number, and attempt to determine how the fluid behaves in the region well outside of the boundary layer. We focus on the Cheng-Minkowycz problem, i.e. the free convective boundary layer flow which is induced by a uniformly hot semi-infinite vertical surface embedded in a porous medium.

**Keywords** Porous media · Boundary layer · Convection · Bingham fluid · Yield stress

# Nomenclature

## Latin letters

$A$	constant
$c$	regularization constant
$f$	reduced streamfunction
$g$	gravity
$G$	threshold body force
$K$	permeability
$L$	length scale
$p$	pressure
Ra	Darcy-Rayleigh number
$T$	temperature (dimensional)
$T_0$	ambient (cold) temperature
$T_1$	temperature of heated surface
$u$	vertical Darcy velocity
$v$	horizontal Darcy velocity
$x$	vertical coordinate
$y$	horizontal coordinate

## Greek letters

$\alpha$	thermal diffusivity
$\beta$	coefficient of cubical expansion
$\eta$	similarity variation
$\theta$	temperature (nondimensional)
$\mu$	dynamic viscosity
$\rho$	reference density
$\sigma$	heat capacity ratio
$\psi$	streamfunction
$\Omega$	scaled threshold body force

## Other symbols

$-$	nondimensional quantities
$\hat{\phantom{x}}$	dimensional quantities
$\sim$	outer region quantities
$'$	derivatives with respect to $\eta$

## 1 Introduction

The aim of this paper is to study the free convection boundary layer flow of a Bingham fluid saturating a porous medium and to determine whether or not these may exist in a domain of infinite extent. A Bingham fluid is one for which there is a yield stress, and thus the fluid moves only when the driving body force, such as a pressure gradient or a buoyancy force or both, is sufficiently great. Once the body force is larger than the yield stress the resulting flow is Newtonian. A Bingham fluid is a special case of a Herschel-Bulkley

fluid for which the stress-strain relationship takes the form,  $\tau = \tau_y + k\dot{\gamma}^n$ , where  $\tau_y$  is the yield stress and  $\dot{\gamma}$  is the rate of strain. The special case of a Bingham fluid corresponds to  $n = 1$ .

There are twelve papers which describe the thermal boundary layer flow of yield-stress fluids in a porous medium. Apart from the paper by Wang and Tu [1], which considers both forced and free convective boundary layers, the governing equations in the remaining works allow for the inclusion of many different effects. For example, Cheng [2] considers variations in the heat mass fluxes at the bounding surface while Cheng [3] extends that analysis by including both the Soret and Dufour effects. Others who have studied double diffusive effects are Jumah and Mujumdar [4,5] and Lakshmi Narayana et al. [6]. Ibrahim et al. [7] consider chemical reactions while the paper by Hady et al. [8] studies nanofluids with a yield stress. Pascal and Pascal [9] analyze the effect of a lateral mass flux and Abdel-Gaied and Eid [10], Wang et al. [11] and Yang and Wang [12] all consider boundary layers which form on axisymmetric bodies. In all cases these authors adopt a boundary layer approximation where  $y$ , the coordinate which is perpendicular to the heated surface, is much less than  $x$ , the coordinate along the surface.

The general aim of this paper is to decide whether the adoption of the boundary layer approximation in an infinite domain yields a consistent flow problem. This aim may seem to be a somewhat unusual thing to do for surely if the local buoyancy force in these twelve papers is sufficiently large to overcome the yield stress of the fluid, thereby giving flow along the heated surface, then why should this be questioned? The reason that it is being questioned may be found in the analysis of Riley and Rees [13]. That paper gives an analysis of free convection boundary layers induced by uniformly hot surfaces, both inclined and horizontal, which are embedded in a porous medium where the fluid is Newtonian. A higher order boundary layer theory was presented there where the flows in both the boundary layer region and the region which is external to the boundary layer (termed the outer region hereinafter) are obtained. Once one has used the classical boundary layer analysis to find the leading order boundary layer solution, then it is possible to find its leading order effect on the isothermal outer region. While the main effect of the heated surface is to induce a strong flow along that surface within the boundary layer, the fluid thus displaced must necessarily be replaced from the outer region in order to set up a circulation wherein replenishing cold fluid is drawn towards the boundary layer. Thus an inflow (or entrainment) exists, and this inflow is then used as a boundary condition for the leading order outer solution. However, the relevant observation from the analysis of Riley and Rees [13] is that this inflow is much weaker in magnitude than that of the buoyancy-induced flow along the heated surface. This is the case for a Newtonian fluid (or even a power-law fluid) without a yield stress. The papers which we have quoted above all have yield stresses which are sufficiently large that there is a yield surface within the boundary layer. Thus streamwise flow ceases outside of that yield surface. If the deduction from the analysis of Riley and Rees [13] is correct, that would also mean that it will not be possible for the weak inflow, which is characteristic of fluids without a yield stress and whose presence is essential for the boundary layer flow to persist, to be strong enough to overcome the yield stress. Therefore the boundary layer cannot be replenished from the external region, and the original boundary layer cannot exist.

In this paper we will carry out a boundary layer analysis which will highlight some of the mathematical difficulties that are associated with a straightforward use of the boundary layer approximation for yield stress fluids when no consideration is given to the effect of the presence of an external region from which fluid is entrained. To this end we consider the Bingham fluid form of the Cheng-Minkowycz problem, which is the steady boundary layer flow which is induced by a uniformly hot semi-infinite surface and, in its Newtonian fluid form, was first studied by Cheng and Minkowycz [14]. This is an exemplar of the difficulties which are mentioned above and this case is analyzed in detail. We close with some brief comments on two different mixed convection problems.

## 2 Governing Equations

Darcy's law for a Newtonian fluid subject to a pressure gradient in the  $\hat{x}$ -direction takes the form,

$$\hat{u} = -\frac{K}{\mu}\hat{p}_{\hat{x}}, \quad (1)$$

where  $K$  is the permeability and  $\mu$  the dynamic viscosity. In this paper we will follow the practice of very many authors and assume that fluid flows in a porous medium only when the body force is larger than the yield value, which we denote by  $\hat{G}$ . For a Bingham fluid, Darcy's law for one-dimensional flow now becomes,

$$\hat{u} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{\hat{G}}{|\hat{p}_{\hat{x}}|}\right]\hat{p}_{\hat{x}} & \text{when } |\hat{p}_{\hat{x}}| > \hat{G}, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

There are more accurate ways of upscaling, such as from an analysis of tube bundles, where the Buckingham-Reiner model may be used:

$$\hat{u} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{4}{3}\left(\frac{\hat{G}}{|\hat{p}_{\hat{x}}|}\right) + \frac{1}{3}\left(\frac{\hat{G}}{|\hat{p}_{\hat{x}}|}\right)^4\right]\hat{p}_{\hat{x}} & \text{if } |\hat{p}_{\hat{x}}| > \hat{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

or its plane-Poiseuille flow equivalent,

$$\hat{u} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{3}{2}\left(\frac{\hat{G}}{|\hat{p}_{\hat{x}}|}\right) + \frac{1}{2}\left(\frac{\hat{G}}{|\hat{p}_{\hat{x}}|}\right)^3\right]\hat{p}_{\hat{x}} & \text{if } |\hat{p}_{\hat{x}}| > \hat{G}, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

For isothermal two-dimensional flows, the frame-invariant form of Eq. (2) is

$$\hat{u} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{\hat{G}}{\sqrt{\hat{p}_{\hat{x}}^2 + \hat{p}_{\hat{y}}^2}}\right]\hat{p}_{\hat{x}} & \text{when } \sqrt{\hat{p}_{\hat{x}}^2 + \hat{p}_{\hat{y}}^2} > \hat{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (5)$$

$$\hat{v} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{\hat{G}}{\sqrt{\hat{p}_{\hat{x}}^2 + \hat{p}_{\hat{y}}^2}}\right]\hat{p}_{\hat{y}} & \text{when } \sqrt{\hat{p}_{\hat{x}}^2 + \hat{p}_{\hat{y}}^2} > \hat{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

Therefore when buoyancy in the vertical direction is also included as an extra body force term, Darcy's law for convection subject to the Boussinesq approximation finally becomes,

$$\hat{u} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{\hat{G}}{\sqrt{[\hat{p}_{\hat{x}} - \rho g \beta(T - T_0)]^2 + \hat{p}_{\hat{y}}^2}}\right](\hat{p}_{\hat{x}} - \rho g \beta(T - T_0)) & \text{when } \sqrt{[\hat{p}_{\hat{x}} - \rho g \beta(T - T_0)]^2 + \hat{p}_{\hat{y}}^2} > \hat{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

$$\hat{v} = \begin{cases} -\frac{K}{\mu}\left[1 - \frac{\hat{G}}{\sqrt{[\hat{p}_{\hat{x}} - \rho g \beta(T - T_0)]^2 + \hat{p}_{\hat{y}}^2}}\right]\hat{p}_{\hat{y}} & \text{when } \sqrt{[\hat{p}_{\hat{x}} - \rho g \beta(T - T_0)]^2 + \hat{p}_{\hat{y}}^2} > \hat{G}, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

The full equations for steady buoyant flow are completed by the equation of continuity,

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} = 0, \quad (9)$$

and the heat transport equation,

$$\hat{u}T_{\hat{x}} + \hat{v}T_{\hat{y}} = \alpha(T_{\hat{x}\hat{x}} + T_{\hat{y}\hat{y}}). \quad (10)$$

In common with most boundary layer analyses we have taken  $\hat{x}$  to be the vertical coordinate. The ambient temperature far from the hot surface is  $T_0$ .

We will work in terms of nondimensional variables so that it is possible to follow a rigorous asymptotic analysis for large values of the Darcy-Rayleigh number. Therefore we rescale the variables according to,

$$(\hat{x}, \hat{y}) = L(\bar{x}, \bar{y}), \quad (\hat{u}, \hat{v}) = \frac{\alpha}{L}(\bar{u}, \bar{v}), \quad \hat{p} = \frac{\alpha\mu}{K}\bar{p}, \quad T = T_0 + (T_1 - T_0)\theta, \quad \hat{G} = \frac{\alpha\mu}{KL}\bar{G}, \quad (11)$$

and then Eqs. (7)–(10) become,

$$\bar{u}_x + \bar{v}_y = 0, \quad (12)$$

$$\bar{u} = \begin{cases} \left[ 1 - \frac{\bar{G}}{\sqrt{(\text{Ra}\theta - \bar{p}_x)^2 + \bar{p}_y^2}} \right] (\text{Ra}\theta - \bar{p}_x) & \text{when } \sqrt{(\text{Ra}\theta - \bar{p}_x)^2 + \bar{p}_y^2} > \bar{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

$$\bar{v} = \begin{cases} - \left[ 1 - \frac{\bar{G}}{\sqrt{(\text{Ra}\theta - \bar{p}_x)^2 + \bar{p}_y^2}} \right] \bar{p}_y & \text{when } \sqrt{(\text{Ra}\theta - \bar{p}_x)^2 + \bar{p}_y^2} > \bar{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

$$\theta_t + \bar{u}\theta_x + \bar{v}\theta_y = \theta_{xx} + \theta_{yy}. \quad (15)$$

In these equations the Darcy-Rayleigh number,  $\text{Ra}$ , is defined according to

$$\text{Ra} = \frac{\rho g \beta (T_1 - T_0) K L}{\mu \alpha}, \quad (16)$$

where  $L$  is a lengthscale. If the flow were to be in a porous channel, for example, then we may take  $L$  to be the width of that channel. If the flow is of boundary layer type in a supposedly infinite domain, then there is no naturally occurring lengthscale. In such cases  $L$  is a so-called fictitious lengthscale, but in practice this means that if we are concentrating our attention on the region near to  $x = 1$  (in nondimensional terms) then this defines the dimensional lengthscale; this is a widespread practice particularly among those authors who use a matched asymptotic expansion to determine the effect of the boundary layer on the external region well outside of the boundary layer and vice-versa.

### 3 Boundary layer analysis

Our aim is perform a standard leading-order boundary layer analysis and to determine its effect on the region outside of the boundary layer. In this section the threshold gradient model as given by Eqs. (5) and (6) will be studied. In the next section a regularised version of these momentum equations will be considered.

We consider Eqs. (12)–(15) for asymptotically large values of  $\text{Ra}$ , and the boundary conditions are taken to be,

$$y = 0 : \quad \theta = 1, \quad v = 0; \quad y \rightarrow \infty : \quad u, \theta \rightarrow 0, \quad (17)$$

which correspond to an impermeable vertical surface which is uniformly hot and to an otherwise cold domain. The boundary layer approximation now needs to be applied to Eqs. (12)–(15) and this done by first determining scalings for the various dependent variables when  $\text{Ra} \gg 1$ . Given our earlier comment about scaling we set  $\bar{x} = O(1)$ , and therefore we have  $\bar{u} = O(\text{Ra})$  from Eq. (13). On balancing the magnitudes of the first nonlinear term and the  $\bar{y}$ -diffusion term in Eq. (15) we find that  $y = O(\text{Ra}^{-1/2})$ ,

and the continuity equation then furnishes  $\bar{v} = O(\text{Ra}^{1/2})$ . Balancing the magnitudes of the two terms in the  $\bar{y}$ -momentum equation yields  $\bar{p} = O(1)$ . Therefore we rescale according to

$$\bar{x} = x, \quad \bar{y} = \text{Ra}^{-1/2}y, \quad \bar{u} = \text{Ra}u, \quad \bar{v} = \text{Ra}^{1/2}v, \quad \bar{p} = p, \quad (18)$$

and consequently Eqs. (12)–(15) reduce to

$$u_x + v_y = 0, \quad (19)$$

$$u = \begin{cases} \left[ 1 - \frac{\bar{G}}{\sqrt{(\text{Ra}\theta - p_x)^2 + \text{Ra}p_y^2}} \right] (\theta - \text{Ra}^{-1}p_x) & \text{when } \sqrt{(\text{Ra}\theta - p_x)^2 + \text{Ra}p_y^2} > \bar{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (20)$$

$$v = \begin{cases} - \left[ 1 - \frac{\bar{G}}{\sqrt{(\text{Ra}\theta - p_x)^2 + \text{Ra}p_y^2}} \right] p_y & \text{when } \sqrt{(\text{Ra}\theta - p_x)^2 + \text{Ra}p_y^2} > \bar{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (21)$$

$$u\theta_x + v\theta_y = \text{Ra}^{-1}\theta_{xx} + \theta_{yy}. \quad (22)$$

Formally the boundary layer approximation now corresponds to letting  $\text{Ra} \rightarrow \infty$ , and this removes the streamwise diffusion term from Eq. (22). Equation (20) now becomes,

$$u = \begin{cases} \theta - \frac{\bar{G}}{\text{Ra}} & \text{when } \text{Ra}\theta > \bar{G}, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

It is clear that when  $\bar{G}$  is of  $O(1)$  magnitude, then the condition for flow is always satisfied while in the  $\text{Ra} \gg 1$  regime, and therefore the boundary layer which is obtained is indistinguishable from that of Cheng and Minkowycz [14] because the flow is essentially Newtonian within the boundary layer.

If we are to achieve a yield surface within the boundary layer, such as been computed by many authors, then  $\bar{G}$  must be of  $O(\text{Ra})$  in magnitude. Therefore we let  $\bar{G} = \text{Ra}\Omega$  and Eq. (23) becomes,

$$u = \begin{cases} \theta - \Omega & \text{when } \theta > \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

Thus the condition for flow to arise is that  $\theta > \Omega$ . Clearly, if  $\Omega > 1$  then there is no induced flow because buoyancy forces are insufficiently strong to overcome the yield criterion and the temperature field must then be  $\theta = 1$  everywhere. Therefore we need  $\Omega$  to lie in the range,  $0 < \Omega < 1$ . In this case it would appear that there will be a flow in that region near to the heated surface where  $\Omega < \theta < 1$ ; it is this type of situation which has been computed by most of the authors cited in the Introduction, but is it attainable? In other words, is it possible to have a boundary layer flow from a uniformly hot surface in a porous medium where there is a yield surface? The detailed answer will follow after we have applied the Cheng-Minkowycz similarity transformation to the governing equations.

First, we introduce the streamfunction in the usual way,

$$u = \psi_y, \quad v = -\psi_x. \quad (25)$$

We will use the scaling,  $\bar{G} = \text{Ra}\Omega$ , as mentioned above. The Cheng-Minkowycz similarity transformation replaces  $y$  by  $\eta = y/x^{1/2}$ . We also set  $\psi = x^{1/2}f(\eta)$ . After some manipulations these transformations yield,

$$f' = \begin{cases} \theta - \Omega & \text{when } \theta > \Omega \\ 0 & \text{otherwise,} \end{cases} \quad (26)$$

and

$$\theta'' + \frac{1}{2}f\theta' = 0, \quad (27)$$

where primes denote differentiation with respect to  $\eta$ . The boundary conditions are that,

$$\eta = 0 : \quad f = 0, \quad \theta = 1; \quad \eta \rightarrow \infty : \quad \theta \rightarrow 0. \quad (28)$$

There is no need to present numerical solutions at this stage because the solution profiles will quite obviously have  $f$  rising from zero to the value it attains when  $\theta = \Omega$ , which we denote by  $f_\infty$ , thereafter continuing as a constant. The temperature field, on the other hand, will decrease from  $\theta = 1$  at the surface and will eventually decay exponentially to zero as a function which is proportional to  $\exp[-f(\infty)\eta/2]$  within the unyielded region. The physical reason why the temperature field decays is because the  $f$ -term in Eq. (27) represents an inflow from outside the boundary layer which brings cooling fluid towards the hot surface. We now consider this inflow.

The velocity components of the boundary layer flow may be shown to be,

$$u = f'(\eta), \quad v = x^{-1/2} [\eta f'(\eta) - f(\eta)]. \quad (29)$$

Therefore, as one exits from the boundary layer itself,  $u \rightarrow 0$ , and hence there is no upward velocity component ( $\bar{u}$ ) of  $O(\text{Ra})$  magnitude in the outer region. At the same time  $v \rightarrow -f_\infty/x^{1/2}$ , which means that there is an inflow ( $\bar{v}$ ) of  $O(\text{Ra}^{1/2})$  magnitude from the outer region into the boundary layer. This inflow must be present in order to maintain the upward flow within the boundary layer. From the point of view of the external domain the boundary layer is very thin, and this inflow may be set formally as a boundary condition for the outer velocity field. Given that  $\theta$  has also now decayed to zero and that we have set  $\bar{G} = \text{Ra}\Omega$ , the full governing momentum equations, Eqs. (20) and (21), reduce to the following form in the outer region,

$$u = \begin{cases} - \left[ 1 - \frac{\text{Ra} \Omega}{\sqrt{p_x^2 + \text{Ra} p_y^2}} \right] \text{Ra}^{-1} p_x & \text{when } \sqrt{p_x^2 + \text{Ra} p_y^2} > \text{Ra} \Omega, \\ 0 & \text{otherwise,} \end{cases} \quad (30)$$

$$v = \begin{cases} - \left[ 1 - \frac{\text{Ra} \Omega}{\sqrt{p_x^2 + \text{Ra} p_y^2}} \right] p_y & \text{when } \sqrt{p_x^2 + \text{Ra} p_y^2} > \text{Ra} \Omega, \\ 0 & \text{otherwise,} \end{cases} \quad (31)$$

where we note that both  $p_x$  and  $p_y$  will be of  $O(1)$  magnitude here. Given that  $\text{Ra} \gg 1$ , it is clear that the inflow velocity is insufficient to overcome the yield criterion since  $\sqrt{p_x^2 + \text{Ra} p_y^2} > \text{Ra} \Omega$  cannot be satisfied.

Therefore we conclude that a rigorous application of boundary layer theory as an asymptotic theory, together with a consideration of whether an external field can exist, has led to the conclusion that  $\bar{G}$  cannot be as large as  $O(\text{Ra})$  since an inflow of  $O(\text{Ra}^{1/2})$  magnitude is required to maintain the upward movement of the fluid due to buoyancy. We could, of course, set  $\bar{G}$  to have an  $O(\text{Ra}^{1/2})$  magnitude; the effect of this is to allow for inflow, but then the buoyancy forces which create the main boundary layer flow are well in excess of the yield threshold, and the resulting boundary layer is then indistinguishable from that of a Newtonian fluid.

## 4 Boundary layer analysis using a regularized model

In this section we will take an alternative approach to determine whether a boundary layer flow of a yield-stress fluid in a porous medium can exist in an infinitely-sized domain. To do this we will relax the concept



of the yield criterion. Thus Eqs. (13) and (14) will be replaced by

$$\left(1 + \overline{G} \frac{\tanh \bar{c} \bar{q}}{\bar{q}}\right) \bar{u} = \text{Ra} \theta - p_x, \quad (32)$$

$$\left(1 + \overline{G} \frac{\tanh \bar{c} \bar{q}}{\bar{q}}\right) \bar{v} = -p_y, \quad (33)$$

where  $\bar{q}^2 = \bar{u}^2 + \bar{v}^2$ . The value  $\bar{c}$  controls how closely the tanh function resembles a yield pressure gradient. To see this, consider the one-dimensional Darcy-Bingham law given by (2) but written in nondimensional form:

$$\bar{u} = \begin{cases} -\left[1 - \frac{\overline{G}}{|\bar{p}_x|}\right] \bar{p}_x & \text{when } |\bar{p}_x| > \overline{G}, \\ 0 & \text{otherwise,} \end{cases} \quad (34)$$

while the unidirectional and isothermal version of (32) is

$$\left(1 + \overline{G} \frac{\tanh \bar{c} \bar{u}}{\bar{u}}\right) \bar{u} = -p_x. \quad (35)$$

Figure 1 shows a schematic of both of these profiles and it is clearly seen that this tanh function gives  $\overline{G}$  as the yield gradient when  $c$  is sufficiently large. For other values of  $c$  we note that  $\tanh \bar{c} \bar{u} \sim \bar{c} \bar{u}$  when  $|\bar{u}|$  is of sufficiently small magnitude, and therefore the effective viscosity for slow flow is that for rapid flow multiplied by the factor,  $(1 + \overline{G} \bar{c})$ . This type of regularization has some physical support from experimental work. For example, Barnes and Walters [15] cite evidence that some fluids possess three different regimes. At high shear rates the viscosity is constant and therefore Newtonian. At very low shear rates the fluid is also Newtonian but the viscosity is very high indeed. Near to what would normally be termed the yield stress, the viscosity changes very rapidly between the two neighbouring constant values. The present tanh-model has the same qualitative features given that the reciprocal of the slope of the tanh-model is proportional to the effective viscosity. Mathematically, the present regularization also follows closely the precedent set by the very common adoption of Papanastasiou's [16] regularization for Bingham fluid flows.

We now follow the same scalings as were introduced in Eq. (18) and set  $\bar{c} = c/\text{Ra}$  for numerical convenience. Thus Eq. (23) is replaced by

$$u + \Omega \tanh cu = \theta. \quad (36)$$

After the introduction of the streamfunction, as given by Eq. (25), then Eqs. (36) and (22) (with the streamwise diffusion term having been neglected because of its order of magnitude), take the following forms,

$$\psi_y + \Omega \tanh(c\psi_y) = \theta, \quad \theta_{yy} = \psi_y \theta_x - \psi_x \theta_y. \quad (37)$$

These equations admit self-similar solutions of the form,  $\psi = x^{1/2} f(\eta)$  and  $\theta = \theta(\eta)$ , where  $\eta$  is again given by  $\eta = y/x^{1/2}$ . Thus we obtain,

$$f' + \Omega \tanh(cf') = \theta, \quad (38)$$

$$\theta'' + \frac{1}{2} f \theta' = 0, \quad (39)$$

and these are to be solved subject to the boundary conditions,

$$f(0) = 0, \quad \theta(0) = 1, \quad \text{and} \quad \theta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty. \quad (40)$$

Solutions of Eqs. (38) and (39) subject to the boundary conditions (40) may be obtained easily using a shooting method program with a fourth order Runge-Kutta solver, for example. Profiles of  $f$  (the gradient of which is proportional to the streamwise velocity) and  $\theta$  are shown in Figure 2 for  $c = 100$  and  $\Omega = 0, 0.1, 0.2, 0.5$  and  $0.8$ . The points which correspond to the yield surface are denoted by small black disks. Given that the large- $\eta$  value of  $f$ , values of which are given in Table 1, corresponds to the total mass flux

up the boundary layer, it is clear that the flow is getting weaker as  $\Omega$  increases which is to be expected. Given the form of Eq. (39), it may be shown that  $\theta = A \exp(-f(\infty)\eta/2)$  and therefore the rate of decay of  $\theta$  towards zero decreases with  $\Omega$ . Thus the thermal boundary layer thickness increases without bound as  $\Omega$  approaches 1 and the consequent reduction in the surface rate of heat transfer may also be found in Table 1.

However, our main aim here is the realizability of boundary layer flows in an infinitely-sized domain. Therefore we will now consider the outer region for the tanh model and then let  $c \rightarrow \infty$  in that solution in order to determine what happens in the yield-threshold model. As a preparation for this, Figure 3 shows the effect of varying  $c$  on the computed profiles for  $f$  and  $\theta$ . In this Figure the pair of profiles which correspond to  $c = 1000$  also correspond essentially to the threshold model. Quantitative data for the variation of  $f(\infty)$  and  $\theta'(0)$  with  $c$  when  $\Omega = 1$  may be found in Table 2.

In the outer region both  $\bar{x}$  and  $\bar{y}$  are of  $O(1)$  magnitude. The streamwise velocity in the boundary layer,  $(\bar{u})$ , has decayed to zero within the  $O(\text{Ra})$  range, while the inflow,  $\bar{v}$ , has  $O(\text{Ra}^{1/2})$  magnitude. If we now assume that both  $\bar{u}$  and  $\bar{v}$  have identical orders of magnitude in the outer region then we may set  $(\bar{u}, \bar{v}) = \text{Ra}^{1/2}(\tilde{u}, \tilde{v})$ . In addition we may set  $\bar{p} = \text{Ra}^{1/2}\tilde{p}$ . Given that  $\bar{c} = c/\text{Ra}$ , the argument to the tanh functions in Eqs. (32) and (33) is asymptotically small, so that we are formally in the high effective viscosity regime. Equations (32) and (33) now become,

$$(1 + \Omega c) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = - \begin{pmatrix} \tilde{p}_x \\ \tilde{p}_y \end{pmatrix}. \quad (41)$$

Apart from the constant coefficient of the velocity vector, this equation was solved in both Daniels and Simpkins [17] and Riley and Rees [13]. In those papers the authors considered the outer region to be a wedge formed by the heated surface being one boundary and an insulated plane surface forming a second boundary. The streamfunction then satisfies Laplace's equation and the solution for the streamfunction depends only on the amplitude of the inflow to the boundary layer, i.e. on the value of  $f(\infty)$ . Figure 3 has already shown that the limit,  $c \rightarrow \infty$ , yields a well-defined boundary layer solution. However, if we now use Eq. (41) to determine the corresponding pressure field, we see that it will be proportional to both  $f(\infty)$  and  $(1 + \Omega c)$ . Therefore the limit,  $c \rightarrow \infty$ , corresponds to infinitely large pressure gradients in the outer region, something which is unphysical. This arises because the effective viscosity of the outer region becomes infinitely large in that limit. We conclude that the large- $c$  limit of the tanh model suggests that a simple threshold model for the flow of a Bingham fluid in a porous medium cannot sustain a boundary layer flow in a domain which is of infinite extent. This will also be true of a Buckingham-Reiner model because it too has a yield threshold, and for Herschel-Bulkley fluids for the same reason.

## 5 Brief comment on mixed convection boundary layers

We now consider briefly the situation which arises for mixed convection boundary layer flows. For simplicity we consider two types: (i) the standard case where there is an external free stream, such as that of a uniform velocity which is parallel to the heated surface; (ii) the case where the heated surface is also a suction surface where fluid is being drawn at a uniform and constant rate into that surface.

In the absence of thermal effects both these types of mixed convection flow are such that the yield threshold has already been exceeded, not just within the boundary layer region, but also within the outer region. For the co-flowing mixed convection problem this means that the difficulties which were studied in §4, which were to do with the impossibility of entrainment in an infinitely-sized domain, now do not arise. Therefore such mixed convection boundary layers will arise. For the suction surface case the resulting boundary layer flow is of uniform thickness and therefore there is no entrainment from the outer region.

The yield threshold now serves only to modify slightly the numerical details of the resulting flow, and there is little point in presenting data because Pascal and Pascal [9] have already considered this flow.

## 6 Conclusions

In this paper we have considered in detail that free convective boundary layer flow of a Bingham fluid in a porous medium which is the Bingham fluid analogue of the Cheng-Minkowycz problem. The main motivation for studying this flow was to determine whether or not previously-published analyses of steady boundary layer flows, ones which consider only the boundary layer itself and not its effect on the outer region, are realizable once the outer flow is considered. The general conclusion is that they are not. We found that the straightforward application of a yield threshold led to an inconsistency in that the required rate of entrainment from the outer region is too small to overcome that yield threshold. A second analysis, which considered a weakening of the yield threshold model by the adoption of a tanh-model, showed that the  $c \rightarrow \infty$  limit causes the pressure field in the outer region to assume unphysically large values, thereby confirming the first analysis. We leave open for the future the question of whether this conclusion also applies to weaker yield thresholds, such as those which are of  $O(\text{Ra}^{1/2})$  in magnitude.

Given our earlier comment, it is of some interest to note that, when  $c$  is finite in the tanh-model, a full steady boundary layer/outer region analysis may be undertaken consistently. The reason that this may be done lies with the fact that the fluid always yields even if the effective viscosity is very large. With this in mind, we note that there are models for both Bingham and Herschel-Bulkley fluids which question the existence a yield stress. For example, Barnes and Walters [15] discuss how, at small shear stresses, there is a regime in which the viscosity decreases as a power law function of the shear stress. Either side of this regime lie what are termed Newtonian plateaux. At high shear rates we have the generally-observed viscosity which is the one that is measured in an experiment. At very low shear rates the fluid is again Newtonian but with an extremely high viscosity. Thus a consistent boundary layer analysis may still be undertaken because the viscosity always remains finite. A true yield threshold is equivalent to having a regime where the viscosity is infinite. Although Barnes [18] called the concept of a yield stress an accurate fiction for the purposes of computation, a similar statement for the specialized area of boundary layer flows is clearly less easy to make in the present context because there is a discontinuous change in the existence properties of the boundary layer as  $c$ , or its equivalent, becomes infinitely large. Our general conclusion, then, is that the strict yield threshold model for a porous medium does not allow for a consistent boundary layer theory when one considers a domain which is of infinite size and which requires entrainment from the outer region.

The two types of mixed convection problem we considered briefly are also realizable in an infinite domain because the agency which causes the forced convection component must also have already exceeded the yield threshold for such a component to exist. The addition of buoyancy forces assists in that regard and there are no yield surfaces. More complicated scenarios, such as a mixed convection flow over a vertically-orientated wedge where the outer flow field takes a power law form near the wedge, are outside the scope of this paper, and would need to be analyzed carefully.

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## Figures

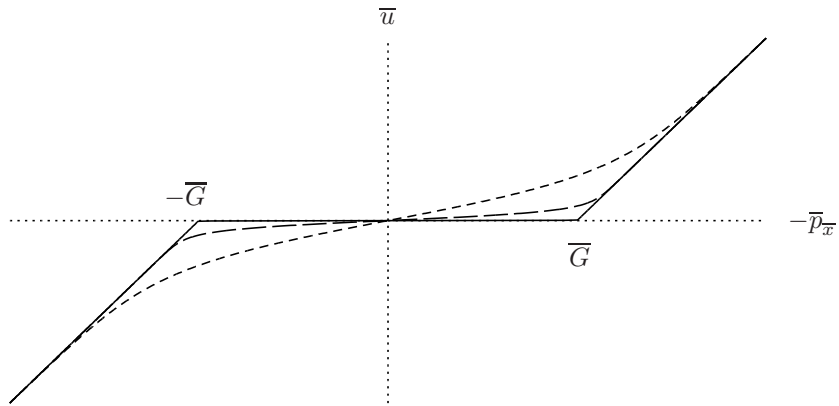


Figure 1: Displaying how the tanh law produces the yield threshold law as  $c$  becomes large. Short dashes correspond to small values of  $\bar{c}$  and long dashes to large values. The threshold model corresponds to the continuous line.

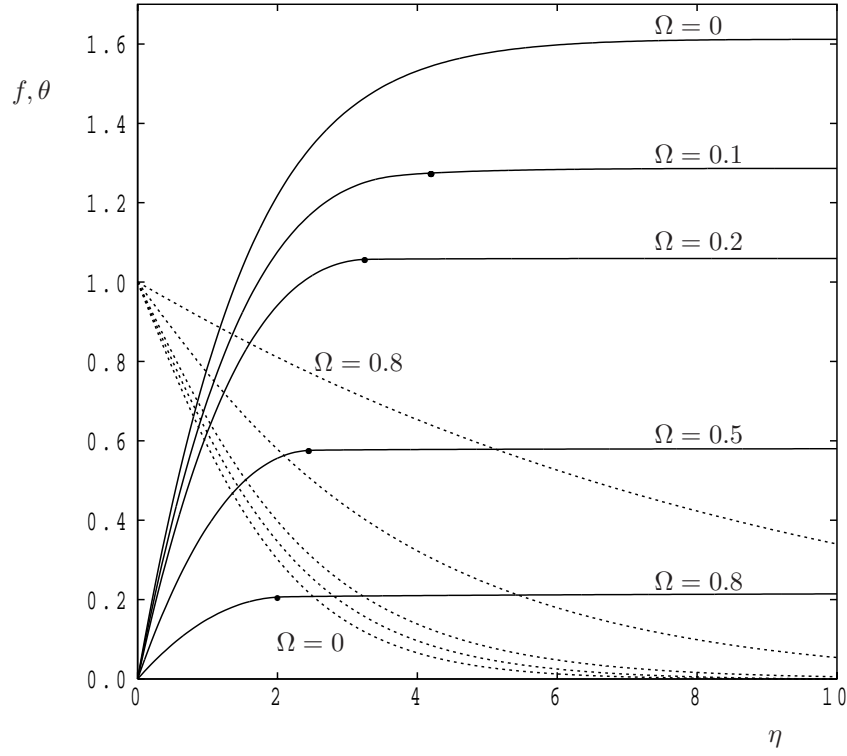


Figure 2: Displaying the profiles of  $f$  (continuous lines) and  $\theta$  (dotted lines) for  $\Omega = 0, 0.1, 0.2, 0.5$  and  $0.8$  and for  $c = 100$ . Yield surfaces are located at the black circles.

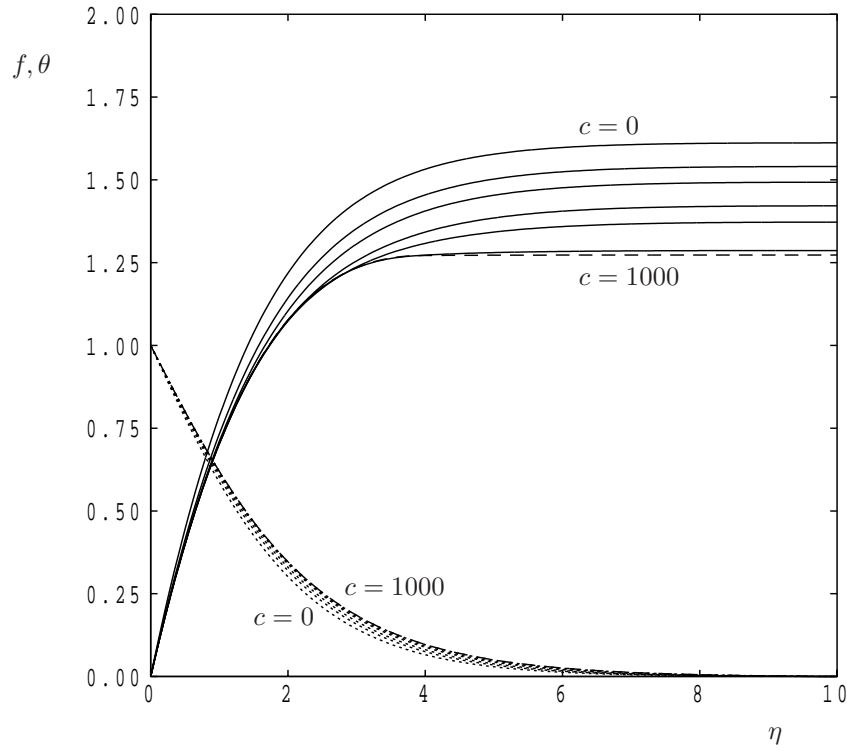


Figure 3: Displaying the profiles of  $f$  (continuous lines) and  $\theta$  (dotted lines) for  $\Omega = 0.1$ , and  $c = 0, 1, 2, 5, 10, 100$  and  $1000$ . Dashed lines correspond to  $c = 1000$ .

$\Omega$	$\eta_{\max}$	$f(\infty)$	$\theta'(0)$
0	20	1.616 125	−0.443 748
0.1	30	1.290 996	−0.405 470
0.2	40	1.076 275	−0.365 725
0.5	70	0.609 411	−0.239 579
0.8	120	0.281 652	−0.109 232

Table 1: Showing the computed values of  $f(\infty)$  and  $\theta'(0)$  for different values of  $\Omega$  when  $c = 100$ . Also shown is the size of the computational domain. The steplength was 0.005 in all cases and all significant figures shown are correct.

$c$	$\eta_{\max}$	$N$	$f(\infty)$	$\theta'(0)$
0	30	6000	1.616 125	−0.443 748
1	30	6000	1.546 007	−0.425 665
2	30	6000	1.499 450	−0.416 714
5	30	6000	1.428 901	−0.409 530
10	30	6000	1.379 521	−0.407 357
$10^2$	30	6000	1.290 996	−0.405 471
$10^3$	30	12000	1.276 118	−0.405 277
$10^4$	30	48000	1.274 512	−0.405 258

Table 2: Showing the computed values of  $f(\infty)$  and  $\theta'(0)$  for different values of  $c$  when  $\Omega = 0.1$ . Also shown are the size of the computational domain and the number,  $N$ , of intervals used in order to have at least six decimal places of accuracy.